South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 309-324

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

COMMON FIXED POINT THEOREM IN COMPLETE METRIC SPACE WITH APPLICATION

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(**Received:** *Nov. 08, 2020* **Accepted:** *Mar. 28, 2021* **Published:** *Apr. 30, 2021)*

Abstract: In this paper, a common fixed point theorem is presented for generalized φ-weak contraction mappings. Also, we examine the existence and uniqueness of common fixed points for single-valued mappings satisfying the notion of weak compatibility in the setup of complete metric space. Our result generalizes and extends many results existing in literature. An example is given to show that our results are proper generalizations of the existing ones and provide an application to integral equation.

Keywords and Phrases: Common fixed point, Complete metric space, Weakly compatible maps.

2020 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

The study of existence and uniqueness of coincidence points and common fixed points of mappings satisfying certain contractive conditions has been an interesting field of mathematics from 1922, when Banach stated and proved his famous result (Banach contraction principle).

Sessa [31] introduced the concept of weakly commuting mappings and obtained some common fixed point theorems in complete metric space. In this way S. T. Patil [24] proved some common fixed point theorems for weakly commuting mappings satisfying a contractive conditions in complete metric space. In 1986, G. Jungck [12] defined compatible mappings and discussed few common fixed point theorems in complete metric space. Also he showed that weak commuting mappings are compatible mappings but converse need not hold. In 1998, the concept of weakly compatible pairs of mappings has been introduced by Jungck [13], that is the class of mappings such that they commute at their coincidence points. In recent years, several authors have obtained coincidence point and common fixed point results for different classes of mappings on various metric spaces such as complete metric spaces, partially ordered metric spaces, Ordered b-metric spaces, b-metric space, cone metric spaces etc see ([1], [8], [10], [17]).

Recently, Parvaneh [22] proved some common fixed point theorems for weakly compatible pair of mapping in the set up of complete metric space. Zhang and Song [35] proved some common fixed point theorems for two single valued generalized φ-weak contraction mappings. Inspired by their work, in the present paper, we prove a common fixed point theorem for generalized φ-weak contraction mappings using the notion of weak compatibility. We justify our theorem by an example and provide an application.

2. Preliminaries

We first present some important definitions. Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and set of real numbers, respectively.

In 1997, Alber and Guerre-Delabriere [3] defined the concept of φ-weak contraction.

Definition 2.1. *A function $T : X \to X$ on a metric space (X,d) is said to be a φ-weak contraction if there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)).$$

Theorem 2.2. [29] *Let (X,d) be a metric space and $T : X \to X$ be a mapping on X such that for all $x, y \in X$, we have*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous and non decreasing function with $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

The following concept of generalized φ-weak contraction was introduced by Zhang and Song [35] in 2009.

Definition 2.3. *Let (X,d) be a metric space. Two self mappings $S, T : X \to X$ are said to be generalized φ-weak contractions if there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,*

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)),$$

where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(y, Tx))\}$.

Theorem 2.4. [35] *Let (X,d) be a metric space and $S, T : X \to X$ be two mappings such that for all $x, y \in X$,*

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)),$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi continuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T and S have a unique common fixed point.

Definition 2.5. [30] *Consider the class of functions $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$, which satisfies the following assertions:*
(i) $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$,
(ii) $(\varphi^n(t))_{n \in N}$ converges to 0 for all $t > 0$,
(iii)$\sum \varphi^n(t)$ converges for all $t > 0$.
If above conditions $(i - ii)$ hold then φ is called a comparison function, and if the comparison function satisfies (iii), then φ is called a strong comparison function.

Remark 2.6. [30] *If $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$ is a comparison function, then $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0.*

3. Main Result

In this section, we establish a common fixed point theorem for weakly compatible mappings.

Theorem 3.1. *Let (X, d) be a metric space and E be a nonempty closed subset of X. Suppose $T, S : E \to E$ are two self mappings and $A, B : E \to X$ are mappings on X such that for all $x, y \in X$,*

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)), \tag{3.1}$$

where

$$M(x, y) = \max\{d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{1}{2}(d(Ax, Sy) + d(By, Tx))\},$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ is lower semi continuous with $0 < \varphi(t) < t$ for $t \in (0, +\infty)$ and $\varphi(0) = 0$ satisfying the following assertions:
(i) $T(E) \subseteq A(E)$ and $S(E) \subseteq B(E)$;
(ii) the pair (T, A) and (S, B) are weakly compatible.
Also, assume that $A(E)$ and $B(E)$ are closed subsets of X. Then T, A, S and B have a unique common fixed point.

Proof. Let $x_0 \in E$. There exists two sequences $\{x_n\}_{n=0}^{+\infty}$ and $\{y_n\}_{n=0}^{+\infty}$ such that $x_{n+1} = Ax_n$ and $y_0 = Tx_0 = Ax_1$, $y_1 = Sx_1 = Bx_2$, $y_2 = Tx_2 = Ax_3$, $\cdots$ $y_{2n} = Tx_{2n} = Ax_{2n+1}$, $y_{2n+1} = Sx_{2n+1} = Bx_{2n+2}$ for all $n \geq 0$. First we proof that $\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0$ from (3.1) we have, $d(y_{2k+1}, y_{2k}) = d(Sx_{2k+1}, Tx_{2k}) = d(Tx_{2k}, Sx_{2k+1})$ which implies

$$d(y_{2k+1}, y_{2k}) \leq M(x_{2k}, x_{2k+1}) - \varphi(M(x_{2k}, x_{2k+1})) \tag{3.2}$$

where

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= \max\{d(Ax_{2k}, Bx_{2k+1}), d(Ax_{2k}, Tx_{2k}), d(Bx_{2k+1}, Sx_{2k+1}), \\ &\quad \frac{1}{2}(d(Ax_{2k}, Sx_{2k+1}) + d(Bx_{2k+1}, Tx_{2k}))\} \\ &= \max\{d(y_{2k-1}, y_{2k}), d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1}), \\ &\quad \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k}))\} \\ &= \max\{d(y_{2k-1}, y_{2k}), d(y_{2k-1}, y_{2k}), \frac{1}{2}(d(y_{2k-1}, y_{2k+1}))\} \\ &= \max\{d(y_{2k-1}, y_{2k}), d(y_{2k-1}, y_{2k}), \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1}))\} \\ &= \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\}. \end{aligned}$$

By (3.2), we have

$$\begin{aligned} d(y_{2k+1}, y_{2k}) &\leq \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\} \\ &\qquad - \varphi[\max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\}] \\ &= d(y_{2k-1}, y_{2k}) \end{aligned}$$

which implies that $d(y_{2k+1}, y_{2k}) \leq d(y_{2k-1}, y_{2k})$. Similarly we can prove that $d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k})$. If $n = 2k + 1$, then $d(y_{n+1}, y_n) \leq d(y_n, y_{n-1})$, for all $n \geq 0$. Thus $\{d(y_n, y_{n+1})\}$ is monotone decreasing sequence of nonnegative real numbers and hence it is convergent. Now

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= M(Bx_{2k-1}, Ax_{2k}) \\ &= M(y_{2k-2}, y_{2k-1}). \end{aligned}$$

Thus $M(y_{2k-2}, y_{2k-1}) = \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\}$.
Suppose $\lim_{n\to+\infty} d(y_{n+1}, y_n) = \lim_{n\to+\infty} M(y_{n-2}, y_{n-1}) = r$. By the lower semi continuity of φ, we have $\varphi(r) = \lim_{n\to+\infty} \inf \varphi(M(y_{n-2}, y_{n-1}))$.
We claim that $r = 0$. From (3.2), we have

$$\begin{aligned} d(y_{n+1}, y_n) &\leq M(x_n, x_{n+1}) - \varphi(M(x_n, x_{n+1})) \\ &= M(Bx_{n-1}, Ax_n) - \varphi(M(Bx_{n-1}, Ax_n)) \\ &= M(y_{n-1}, y_n) - \varphi(M(y_{n-1}, y_n)). \end{aligned}$$

Taking limit as $n \to +\infty$ on the above inequality, we have

$$r \leq r - \varphi(r) \quad \textit{implies} \quad \varphi(r) \leq 0.$$

Thus $\varphi(r) = 0$, by the property of the function φ. Hence

$$\lim_{n\to+\infty} d(y_{n+1}, y_n) = r = 0. \tag{3.3}$$

Next, we show that $\{y_n\}$ is a Cauchy sequence. We know that $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, it is sufficient to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that $d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$ for $n(k) > m(k) > k$. This means that $d(y_{2m(k)}, y_{2n(k)-1}) < \epsilon$. From triangle inequality,

$$\begin{aligned} \epsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &\leq \epsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}). \end{aligned}$$

Letting $k \to +\infty$ in the above inequality, we have

$$\lim_{k\to+\infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon. \tag{3.4}$$

Moreover

$$|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1}) \tag{3.5}$$

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) \tag{3.6}$$

$$|d(y_{2n(k)}, y_{2m(k)-2}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2m(k)-2}, y_{2m(k)-1}). \tag{3.7}$$

From the equations (3.3), (3.4), (3.5), (3.6) and (3.7) we conclude that

$$\begin{aligned}\lim_{n\to+\infty} d(y_{2m(k)-1}, y_{2n(k)}) &= \lim_{n\to+\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) \\ &= \lim_{n\to+\infty} (y_{2m(k)-2}, y_{2n(k)}) \\ &= \epsilon. \end{aligned} \tag{3.8}$$

Now from (3.1), we have

$$\begin{aligned} d(y_{2m(k)-1}, y_{2n(k)}) &= d(Sx_{2m(k)-1}, Tx_{2n(k)}) \\ &= d(Tx_{2n(k)}, Sx_{2m(k)-1}) \\ &= M(x_{2n(k)}, x_{2m(k)-1}) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})). \end{aligned} \tag{3.9}$$

Here

$$\begin{aligned} M(x_{2n(k)}, x_{2m(k)-1}) &= \max\{d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}), \\ &\quad d(Bx_{2m(k)-1}, Sx_{2m(k-1)}), \\ &\quad \frac{1}{2}(d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Bx_{2m(k)-1}, Tx_{2n(k)}))\} \\ &= \max\{d(y_{2n(k)-1}, y_{2m(k)-2}), d(y_{2n(k)-1}, y_{2n(k)}), \\ &\quad d(y_{2m(k)-2}, y_{2m(k)-1}), \\ &\quad \frac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)}))\}. \end{aligned}$$

Now, we consider the following cases:
Case I. If $M(x_{2n(k)}, x_{2m(k)-1}) = d(y_{2n(k)-1}, y_{2m(k)-2})$, then taking limit as $n \to +\infty$ in (3.9), we get $\epsilon \le \epsilon - \varphi(\epsilon) \Rightarrow \varphi(\epsilon) = 0$. By our assumption about φ, we have $\epsilon = 0$, which is a contradiction.
Case II. If $M(x_{2n(k)}, x_{2m(k)-1}) = d(y_{2n(k)-1}, y_{2n(k)})$, then taking limit as $n \to +\infty$ in (3.9), we get, $\epsilon \le 0 - \varphi(0)$, which is a contradiction.
Case III. Finally, if $M(x_{2n(k)}, x_{2m(k)-1}) = \frac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)}))$, then taking limit as $n \to +\infty$ in (3.9), we get $\epsilon \le \frac{1}{2}(\epsilon + \epsilon) - \varphi(\frac{1}{2}(\epsilon + \epsilon))$, or $\epsilon \le \epsilon - \varphi(\epsilon)$, which is a contradiction.
Hence $\{y_n\}$ must be a Cauchy sequence.

Finally we show that T, S, A and B have a common fixed point. Since (X, d) is complete and $\{y_n\}$ is a Cauchy sequence in X, there is a $z \in X$ such that $\lim_{n\to+\infty} y_n = z$. Also, being a closed subset of X, E is complete and $\{y_n\} \subset E$; We have $z \in E$. By assumption $A(E)$ is closed, there exists $u \in E$ such that $z = Au$. For all $n \in N$.

$$d(Tu, y_{2n+1}) = d(Tu, Sx_{2n+1}) \le M(u, x_{2n+1}) - \varphi(M(u, x_{2n+1})), \tag{3.10}$$

Now,

$$\begin{aligned}
M(u, x_{2n+1}) &= \max\{d(Au, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\quad \frac{1}{2}(d(Au, Sx_{2n+1}) + d(Bx_{2n+1}, Tu))\} \\
&= \max\{d(z, y_{2n+1}), d(z, Tu), d(y_{2n}, y_{2n+1}), \\
&\quad \frac{1}{2}(d(z, y_{2n+1}) + d(y_{2n}, Tu))\}.
\end{aligned}$$

Taking limit $n \to +\infty$ in (3.10), we get $d(Tu, z) = 0$. So, $Tu = z$. Similarly we can show that $Su = z$. Therefore $Tu = Su = Au = Bu = z$.
Since the pair (A, T) and (B, S) are weakly compatible, we have $Tz = Sz = Az = Bz$. Therefore

$$\begin{aligned}
d(Tz, y_{2n+1}) &= d(Tz, S_{x2n+1}) \\
&\le M(z, x_{2n+1}) - \varphi(M(z, x_{2n+1})).
\end{aligned} \tag{3.11}$$

Here

$$\begin{aligned}
M(z, x_{2n+1}) &= \max\{d(Az, Bx_{2n+1}), d(Az, Tz), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\quad \frac{1}{2}(d(Az, Sx_{2n+1}) + d(Bx_{2n+1}, Tz))\} \\
&= \max\{d(z, y_{2n+1}), d(z, Tz), d(y_{2n}, y_{2n+1}), \\
&\quad \frac{1}{2}(d(z, y_{2n+1}) + d(y_{2n}, Tz))\}.
\end{aligned}$$

Taking limit $n \to +\infty$ in (3.11), we get $d(Tz, z) = 0$. So $Tz = z$. Similarly we can show that $Sz = z$. Therefore $Tz = Sz = Az = Bz$, we conclude that $Tz = Sz = Az = Bz = z$. Thus, z is a common fixed point of T, S, A and B.
If there exists another point $v \in E$ such that $v = Tv = Sv = Av = Bv$, then using similar argument, we get

$$\begin{aligned}
d(z, v) &= d(Tz, Sv) \\
&= M(z, v) - \varphi(M(z, v)) \\
&= d(z, v) - \varphi(d(z, v))).
\end{aligned}$$

Hence $z = v$. Therefore T, S, A and B have a unique common fixed point.

By the virtue of Theorem 3.1, we easily deduce the following well known results as corollaries.

If the mapping S=T in Theorem 3.1, we have the following.

Corollary 3.1. *Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $T : E \to E$ be self map such that for all $x, y \in E$*

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)), \tag{3.12}$$

where $M(x, y) = \max d(Ax, By), d(Ax, Tx), d(By, Ty), \frac{1}{2}(d(Ax, Ty) + d(By, Tx))$; $\varphi : [0, +\infty) \to [0, +\infty)$ is lower semi continuous with $0 < \varphi(t) < t$ for $t \in (0, +\infty)$ and $\varphi(0) = 0$ and $A, B : E \to X$ satisfying the following assertions:
(i) $T(E) \subseteq A(E)$ and $T(E) \subseteq B(E)$;
(ii) the pair (T, A) and (T, B) are weakly compatible. Also, assume that $A(E)$ is a closed subset of X. Then T, A and B have a unique common fixed point.

If $S = T$ and $A = B$ in theorem 3.1, we have the following.

Corollary 3.2. *Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $T : E \to E$ be self map such that for all $x, y \in E$*

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)), \tag{3.13}$$

where $M(x, y) = \max d(Ax, Ay), d(Ax, Tx), d(Ay, Ty), \frac{1}{2}(d(Ax, Ty) + d(Ay, Tx))$. $\varphi : [0, +\infty) \to [0, +\infty)$ is lower semi continuous with $0 < \varphi(t) < t$ for $t \in (0, +\infty)$ and $\varphi(0) = 0$ and $A : E \to X$ satisfying the following assertions:
(i) $T(E) \subseteq A(E)$;
(ii) the pair (T, A) are weakly compatible.
Also, assume that $A(E)$ is a closed subset of X. Then T and A have a unique common fixed point.

If $A = B$ we get the main result of [19] as corollary.

Corollary 3.3. *Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $T, S : E \to E$ be self mappings (at least one of T or S is surjective) such that for all $x, y \in E$*

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)),$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is lower semi continuous with $\varphi(t) > 0$ for $t \in (0, +\infty)$ and $\varphi(0) = 0$ and $A : E \to X$ satisfying the following assertions:
(i) $T(E) \subset A(E)$ and $S(E) \subset A(E)$.
(ii) The pair (T, A) and (S, A) are weakly compatible.
Also, assume that $A(E)$ is a closed subset of X. Then T, A and S have a unique common fixed point.

If A and B are identity mappings and we take $\varphi(t) = k(t)$, where $k \in [0, 1)$ then

Corollary 3.4. *Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $S, T : E \to E$ be self mappings and satisfy the condition of Theorem 3.1.*

$$\begin{aligned} d(Tx, Sy) &\leq (1-k)M(x, y) \\ &\leq \alpha M(x, y) \qquad \alpha \in (0, 1] \end{aligned}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(y, Tx))\}$.

Now we furnish an example to demonstrate the validity of Theorem 3.1.

4. Example

Example 4.1. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a complete metric space. Let $E = [0, 1]$. The self maps $T, S : E \to E$ are defined as $T(0) = 0$, $T(x) = \frac{1}{x+6}(x \neq 0)$ and $S(x) = \frac{x^2}{4}$, for all $x \in E$. Let the mappings $A, B : E \to X$ and $\varphi : [0, +\infty] \to [0, +\infty]$ be defined by $A(x) = x$, and $B(x) = \frac{x}{2}$ for all $x \in E$, and $\varphi(t) = \frac{t^3}{5}$.

Solution. Clearly,$T(E) \subset A(E)$ and $S(E) \subset A(E)$, which satisfy the condition (i) of theorem 3.1 and $T(A(0)) = A(T(0)) = 0$ and $S(B(0)) = B(S(0)) = 0$, we see that A and T as well as B and S are weakly compatible since they commute as their coincidence point $x = 0$. $T(1) = \frac{1}{7}$, $S(\frac{1}{2}) = \frac{1}{16}$, $A(1) = 1$, $B(\frac{1}{2}) = \frac{1}{4}$. By the equation (3.1), we have $d(T(x), S(y)) = d(T(1), S(\frac{1}{2})) = |\frac{1}{7} - \frac{1}{16}| = \frac{9}{112}$ and

$$\begin{aligned} M(1, \frac{1}{2}) &= \max\{d(A(1), B(\frac{1}{2})), d(A(1), T(1), d(B(\frac{1}{2}), S(\frac{1}{2})), \\ &\quad \frac{1}{2}(d(A(1), S(\frac{1}{2})) + d(B(\frac{1}{2}), T(1)))\} \\ &= \max\{d\big(1, \frac{1}{4}\big), d\big(1, \frac{1}{7}\big), d\big(\frac{1}{4}, \frac{1}{16}\big), \frac{1}{2}\big(d\big(1, \frac{1}{16}\big) + d\big(\frac{1}{4}, 1\big)\big)\} \\ &= \max\big\{\frac{3}{4}, \frac{6}{7}, \frac{3}{16}, \frac{27}{32}\big\} \\ &= \frac{6}{7} \end{aligned}$$

there for,

$$\begin{aligned} M(x, y) - \varphi(M(x, y)) &= \frac{6}{7} - \varphi(\frac{6}{7}) \\ &= \frac{294}{343} \end{aligned}$$

Hence $d(T(x), S(y)) \leq M(x, y) - \varphi(M(x, y))$.
Here, we see that all the conditions of Theorem 3.1 are satisfied. This implies that T, S, A and B have a unique common fixed point ($x = 0$).

5. Application

Consider the following pair of nonlinear integral equations:

$$x(t) = f_1(t) - f_2(t) + \mu \int_a^t m(t,s) g_i(s, x(s)) ds + \lambda \int_a^{+\infty} k(t,s) h_j(s, x(s)) ds \quad (5.1)$$

for all $t \in [a, +\infty)$, where f_1, $f_2 \in L[a, +\infty)$ are known, $f_1(t) \geq f_2(t)$, $m(t,s)$, $k(t,s)$, $g_i(s, x(s))$, $h_j(s, y(s))$, i, j = 1, 2 and $i \neq j$ are real or complex valued functions that are measurable both in t and s on $[a, +\infty)$ and λ, μ are real or complex numbers. These functions satisfy the following:
(C1) $\int_a^{+\infty} \sup_{a<s<+\infty} |m(t,s)| dt = M_1 < +\infty$
(C2) $\int_a^{+\infty} \sup_{a<s<+\infty} |k(t,s)| dt = M_2 < +\infty$
(C3) $g_i(s, x(s)) \in L[a, +\infty)$ for all $x \in L[a, +\infty)$ and there exists $K_1 > 0$ such that for all $s \in [a, +\infty)$,

$$|g_1(s, x(s)) - g_2(s, y(s))| \leq K_1 |x(s) - y(s)| \quad for \ \ all \ \ x, y \in L[a, +\infty)$$

(C4) $h_i(s, x(s)) \in L[a, +\infty)$ for all $x \in L[a, +\infty)$ and there exists $K_2 > 0$ such that for all $s \in [a, +\infty)$,

$$|h_1(s, x(s)) - h_2(s, y(s))| \leq K_2 |x(s) - y(s)| \quad for \ \ all \ \ x, y \in L[a, +\infty).$$

The existence theorem can be formulated as follows:

Theorem 5.1. *With the assumption* $(C1) - (C4)$, *if the following conditions are also satisfied:*
(a) $\lambda \int_a^{+\infty} k(t,s) h_i(s, \mu \int_a^s m(s,\tau) g_j(\tau, x(\tau)) d\tau + f_1(s) - f_2(s)) ds = 0$, $i, j = 1, 2$, $i \neq j$.
(b) For some $x \in L[a, +\infty)$,

$$\begin{aligned}\mu \int_a^t m(t,s) g_i(s, x(s)) ds &= x(t) - f_1(t) + f_2(t) - \lambda \int_a^{+\infty} k(t,s) h_i(s, x(s)) ds \\ &= \Gamma_i(t) \in L[a, +\infty), \quad i = 1, 2.\end{aligned}$$

(c) If for some $\Gamma_i(t) \in L[a, +\infty)$ *there exist* $\Theta_i(t) \in L[a, +\infty)$, *such that*

$$\begin{aligned}\mu \int_a^t m(t,s) g_i(s, x(s) - \Gamma_i(s)) ds - f_2(t) &= f_1(t) + \lambda \int_a^{+\infty} k(t,s) h_i(s, x(s) - \Gamma_i(s) - f_2(s)) ds \\ &= \Theta_i(t), \quad i = 1, 2\end{aligned}$$

then the system (5.1) has a unique solution in $L[a,+\infty)$ *for each pair of real or complex numbers* λ *and* μ *with* $|\lambda|K_2M_2 < 1$ *and* $\frac{|\mu|K_1M_1}{1-|\lambda|K_2M_2} = \alpha(say) < 1$.

Proof. We define, for every $x \in L[a,+\infty)$

$$Sx(t) = \mu \int_a^t m(t,s)g_1(s,x(s))ds - f_2(t)$$

$$Tx(t) = \mu \int_a^t m(t,s)g_2(s,x(s))ds - f_2(t)$$

$$Ux(t) = f_1(t) + \lambda \int_a^{+\infty} k(t,s)h_1(s,x(s))ds$$

$$Vx(t) = f_1(t) + \lambda \int_a^{+\infty} k(t,s)h_2(s,x(s))ds$$

$Ax(t) = (I-U)x(t)$, $Bx(t) = (I-V)x(t)$,
where f_1, $f_2 \in L[a,+\infty)$ are known and I is the identity mappings on $L[a,+\infty)$. Then S, T, U, V, A and B are mappings from $L[a,+\infty)$ into itself.
Indeed, we have:

$$\begin{aligned}|S(x)| &\le |\mu| \int_a^{+\infty} |m(t,s)g_1(s,x(s))|ds + |f_2(t)| \\ &\le |\mu| \sup_{a<s<+\infty} |m(t,s)| \int_a^{+\infty} |g_1(s,x(s))|ds + |f_2(t)|,\end{aligned}$$

we apply conditions $(C1)$ and $(C2)$ in above inequality and thus we have

$$\begin{aligned}\int_a^{+\infty} |S(x)|dt &\le |\mu| \int_a^{+\infty} \sup_{a<s<+\infty} |m(t,s)|dt \int_a^{+\infty} |g_1(s,x(s))|ds + \int_a^{+\infty} |f_2(t)|dt \\ &< +\infty, \end{aligned} \tag{5.2}$$

and hence $Sx \in L[a,+\infty)$. Similarly $Tx \in L[a,+\infty)$ also. For mapping U, we apply the conditions $(C2)$ and $(C4)$ in the following manner:

$$\begin{aligned}\int_a^{+\infty} |Ux(t)|dt &\le \int_a^{+\infty} |f_1(t)|dt + |\lambda| \int_a^{+\infty} \sup_{a<s<+\infty} |k(t,s)|dt \int_a^{+\infty} |h_1(s,x(s))|ds \\ &< +\infty. \end{aligned} \tag{5.3}$$

As $f \in L[a,+\infty)$ and so mapping U is a self mapping on $L[a,+\infty)$. Similarly V is also a self mapping on $L[a,+\infty)$. With the help of $(C2)$ and $(C3)$, we have for all $x, y \in L[a,+\infty)$

$$\begin{aligned}
|Sx - Ty| &= \int_a^{+\infty} |Sx(t) - Tx(t)|dt \\
&= \int_a^{+\infty} \Big|\mu \int_a^t m(t,s)g_1(s,x(s))ds - \mu \int_a^t m(t,s)g_2(s,y(s))ds\Big|dt \\
&= \int_a^{+\infty} \Big|\mu \int_a^t m(t,s)[g_1(s,x(s)) - g_2(s,y(s))]ds\Big|dt \\
&\le \int_a^{+\infty} |\mu| \sup_{a<s<+\infty} |m(t,s)|dt \int_a^{+\infty} |[g_1(s,x(s))g_2(s,y(s))]|ds \\
&= |\mu|K_1M_1 \int_a^{+\infty} |x(s) - y(s)|ds \\
&\le |\mu|K_1M_1|x-y| \\
&\le |\mu|K_1M_1 \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(x,Sy) + d(y,Tx)]\} \\
&\le \alpha M(x,y). \qquad (5.4)
\end{aligned}$$

Similarly, by $(C2)$ and $(C4)$, we get

$$\begin{aligned}
|Ux - Vy| &\le |\lambda|K_2M_2|x-y| \\
&= |\lambda|K_2M_2 \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(x,Sy) + d(y,Tx)]\}. \qquad (5.5)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
|Ax - Bx| &= |(I-U)x - (I-V)y| \\
&= |(x-y) - (Ux - Vy)| \\
&\ge |x-y| - |Ux - Vy| \\
&\ge |x-y| - |\lambda|K_2M_2|x-y| \\
&\ge (1 - |\lambda|K_2M_2)|x-y|, \\
&\ge (1 - |\lambda|K_2M_2) \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(x,Sy) + d(y,Tx)]\} \qquad (5.6)
\end{aligned}$$

which implies

$$|x-y| \le \frac{1}{1 - |\lambda|K_2M_2}|Ax - By|. \qquad (5.7)$$

From (5.4) and (5.7), we obtain

$$\begin{aligned}
|Sx - Ty| &\leq |\mu| K_1 M_1 |x - y| \\
&\leq |\mu| K_1 M_1 \frac{1}{1 - |\lambda| K_2 M_2} |Ax - By| \\
&\leq \frac{|\mu| K_1 M_1}{1 - |\lambda| K_2 M_2} |Ax - By| \\
&\leq \alpha |Ax - By| \\
&\leq \alpha |x - y| \qquad (if A = B = I) \\
&\leq \alpha \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(x, Sy) + d(y, Tx)]\}. \qquad (5.8)
\end{aligned}$$

Thus all the condition of Corollary 3.4 are satisfied, there exist a unique $u \in L(a, +\infty)$ such that $Sv = Tv = v$, and consequently, v is the unique solution of (5.1).

6. Conclusion

The purpose of this paper is to give the generalized version of φ-weak contraction mappings on complete metric space. Also, we derive common coupled fixed point theorems in complete metric spaces. Some examples are presented to verify the effectiveness and applicability of our main results. Towards the end, an application to integral equations has also been presented to support the usability of the obtained results.

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